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Some new exact solutions of the Novikov–Veselov equation

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Abstract. The Novikov–Veselov (NV) equation is considered. Using Bäcklund transformations and a nonlinear superposition formula for the NV equation, some new exact solutions to this equation are found, among which are the so-called one- and two-lump solutions.

It is always an interesting topic to search for new exact solutions of partial differential equations, especially for those which appear in many branches of nonlinear science. In this paper, we would like to search for some new exact solutions of the so-called Novikov–Veselov equation. This equation [1, 2] can be written as follows

$$2u_t + u_{xxx} + u_{yyy} + 3(u\partial_y^{-1}u_x)_x + 3(u\partial_x^{-1}u_y)_y = 0 \tag{1}$$

and may be viewed as a version in two spatial dimensions and one temporal dimension of the well known Korteweg–de Vries (KdV) equation. Much research on this equation has been conducted [1–9]. For example, the NV equation is solvable via inverse scattering techniques (IST) (see, e.g., [3]) and has a Bäcklund transformation, Moutard transformation and soliton-like solutions [6–8]. Recently, a nonlinear superposition formula for the NV equation was proved under certain conditions by one of the present authors [9] and some particular solutions were given as an application of the obtained result.

Let us introduce the following dependent variable transformation

$$u = u_0 + 2(\ln f)_{xy}$$

where u_0 is a constant.

In this case, the NV equation (1) may be rewritten [6, 9] as

$$D_x[(D_x^3D_y + 2D_tD_y + 3u_0D_x^2)f \cdot f] \cdot f^2 + D_y[(D_xD_y^3 + 3u_0D_y^2)f \cdot f] \cdot f^2 = 0 \tag{2}$$

where the bilinear operator $D_x^\ell D_t^m D_y^n$ is defined by $D_x^\ell D_t^m D_y^n a(x,t,y) \cdot b(x,t,y) \equiv (\partial_x - \partial_{x'})^\ell(\partial_t - \partial_{t'})^m(\partial_y - \partial_{y'})^n a(x,t,y)b(x',t',y')|_{x'=x,t'=t,y'=y}$. A Bäcklund transformation (BT) for (2) is given by [6, 9]

$$(D_xD_y - \mu D_x - \lambda D_y + \lambda\mu + u_0)f \cdot f' = 0 \tag{3}$$

$$(2D_t + D_x^3 + D_y^3 + 3\lambda^2D_x - 3\lambda D_x^2 + 3\mu^2D_y - 3\mu D_y^2)f \cdot f' = 0 \tag{4}$$

where λ and μ are arbitrary constants.

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We now choose $\mu = -u_0/\lambda$. In this case, (3, 4) becomes

$$\left(D_x D_y + \frac{u_0}{\lambda} D_x - \lambda D_y\right) f \cdot f' = 0 \tag{5}$$

$$\left(2D_t + D_x^3 + D_y^3 + 3\lambda^2 D_x - 3\lambda D_x^2 + 3\frac{u_0^2}{\lambda^2} D_y + 3\frac{u_0}{\lambda} D_y^2\right) f \cdot f' = 0. \tag{6}$$

We represent the BT (5) and (6) symbolically by $f \xrightarrow{\lambda} f'$. In the following, under certain conditions, we shall establish a nonlinear superposition formula for equation (2). To this end let f_0 be a solution of equation (2), $f_0 \neq 0$. Suppose that f_i ($i = 1, 2$) is a solution of (2) which is related to f_0 under the BT (5) and (6) with parameter λ_i, i.e. $f_0 \xrightarrow{\lambda_i} f_i$ ($i = 1, 2$), and that a new function f_{12} is defined by the superposition formula:

$$[D_x - (\lambda_1 + \lambda_2)] f_0 \cdot f_{12} = [D_x + (\lambda_1 - \lambda_2)] f_1 \cdot f_2. \tag{7}$$

From these assumptions and in analogy to the deduction given in [9], we can prove that f_{12} is a new solution of equation (2) and that it is related to f_1 and f_2 by the BT (5) and (6) (i.e. $f_1 \xrightarrow{\lambda_2} f_{12}$, $f_2 \xrightarrow{\lambda_1} f_{12}$) provided that it satisfies the following two constraints:

$$D_y f_1 \cdot f_2 + \left(\frac{u_0}{\lambda_2} - \frac{u_0}{\lambda_1}\right) f_1 f_2 + D_y f_0 \cdot f_{12} + \left(\frac{u_0}{\lambda_2} + \frac{u_0}{\lambda_1}\right) f_0 f_{12} = 0 \tag{8}$$

$$\begin{aligned} &2D_t f_0 \cdot f_{12} + \tfrac{1}{4} D_y^3 f_0 \cdot f_{12} + \tfrac{1}{4} D_x^3 f_0 \cdot f_{12} + \tfrac{3}{2}(\lambda_1^2 + \lambda_2^2) D_x f_0 \cdot f_{12} \\ &\quad - \tfrac{3}{4}(\lambda_1 + \lambda_2) D_x^2 f_0 \cdot f_{12} + \frac{3}{2}\left(\frac{u_0^2}{\lambda_1^2} + \frac{u_0^2}{\lambda_2^2}\right) D_y f_0 \cdot f_{12} \\ &\quad + \frac{3}{4}\left(\frac{u_0}{\lambda_1} + \frac{u_0}{\lambda_2}\right) D_y^2 f_0 \cdot f_{12} - \tfrac{3}{4} D_y^3 f_1 \cdot f_2 + \tfrac{3}{4} D_x^3 f_1 \cdot f_2 \\ &\quad + \frac{9}{4}\left(\frac{u_0}{\lambda_1} - \frac{u_0}{\lambda_2}\right) D_y^2 f_1 \cdot f_2 + \tfrac{9}{4}(\lambda_1 - \lambda_2) D_x^2 f_1 \cdot f_2 \\ &\quad + \tfrac{3}{2}(\lambda_1 - \lambda_2)^2 D_x f_1 \cdot f_2 - \frac{3}{2}\left(\frac{u_0}{\lambda_1} - \frac{u_0}{\lambda_2}\right)^2 D_y f_1 \cdot f_2 = 0. \end{aligned} \tag{9}$$

One can use the above superposition formula (7) to generate new particular solutions of equation (2). It is, for example, easily verified that 1 and $\theta_i + \alpha_i \equiv \lambda_i^2 x + u_0 y - 3/2(\lambda_i^4 + (u_0^3/\lambda_i^2))t + \alpha_i$ are solutions of equation (2), linked to each other by the BT (5) and (6): $1 \xrightarrow{\lambda_i} \theta_i + \alpha_i$, where α_i ($i = 1, 2$) is a constant. For this ansatz, one can calculate from formula (7) that

$$\begin{aligned} f_{12} = {} & \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}\theta_1\theta_2 + \left[\frac{2\lambda_1\lambda_2^2}{(\lambda_1 + \lambda_2)^2} - \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\alpha_2\right]\theta_1 - \left[\frac{2\lambda_1^2\lambda_2}{(\lambda_1 + \lambda_2)^2} + \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\alpha_1\right]\theta_2 \\ & + \frac{2\lambda_1^2\lambda_2^2(\lambda_2 - \lambda_1)}{(\lambda_1 + \lambda_2)^3} + \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}\alpha_1\alpha_2 + 2\lambda_1\lambda_2\frac{\alpha_1\lambda_2 - \alpha_2\lambda_1}{(\lambda_1 + \lambda_2)^2}. \end{aligned} \tag{10}$$

It can be easily shown, by direct calculation, that this particular f_{12} satisfies the conditions (8) and (9). Therefore f_{12} given by expression (10) is a new solution of equation (2). In particular, if we choose u_0 to be real, λ_1 and λ_2 to be complex conjugate ($\lambda_1 = a + b\sqrt{-1}$ (a, b real), $\lambda_2 = \lambda_1^*$) and

$$\alpha_1 = -\frac{2\lambda_1^2\lambda_2}{\lambda_1^2 - \lambda_2^2} \qquad \alpha_2 = \frac{2\lambda_1\lambda_2^2}{\lambda_1^2 - \lambda_2^2}$$

we have the following solution for equation (2):

$$f = \theta_1\theta_1^* + \frac{4\lambda_1^3\lambda_2^3}{(\lambda_1^2-\lambda_2^2)^2} + \frac{2\lambda_1^2\lambda_2^2}{(\lambda_1+\lambda_2)^2} = \theta_1\theta_1^* + \frac{b^4-a^4}{4a^2b^2}(a^2+b^2). \tag{11}$$

Hence, when $b^2 > a^2$ ($ab \neq 0$), formula (11) represents a one-lump solution for equation (2). In general, along the same lines, we can obtain more explicit solutions of equation (2). For example, we have

$$\begin{array}{ccccccccc} & & f_2 & & & & f_{123} & & \\ & {\scriptstyle\lambda_2}\nearrow & & \searrow{\scriptstyle\lambda_3} & & {\scriptstyle\lambda_1}\nearrow & & \searrow{\scriptstyle\lambda_4} & \\ f_0 & & & & f_{23} & & & & f_{1234} \\ & {\scriptstyle\lambda_3}\searrow & & \nearrow{\scriptstyle\lambda_2} & & {\scriptstyle\lambda_4}\searrow & & \nearrow{\scriptstyle\lambda_1} & \\ & & f_3 & & & & f_{234} & & \end{array}$$

where f_{123}, f_{234} and f_{1234} are polynomials of x, y, t of the forms

$$\begin{aligned} f_{123} &= A_1\theta_1\theta_2\theta_3 + A_2\theta_1\theta_2 + A_3\theta_2\theta_3 + A_4\theta_1\theta_3 + A_5\theta_1 + A_6\theta_2 + A_7\theta_3 + A_8 \\ f_{234} &= B_1\theta_2\theta_3\theta_4 + B_2\theta_2\theta_3 + B_3\theta_3\theta_4 + B_4\theta_2\theta_4 + B_5\theta_2 + B_6\theta_3 + B_7\theta_4 + B_8 \\ f_{1234} &= C_1\theta_1\theta_2\theta_3\theta_4 + C_2\theta_1\theta_2\theta_3 + C_3\theta_1\theta_2\theta_4 + C_4\theta_1\theta_3\theta_4 + C_5\theta_2\theta_3\theta_4 + C_6\theta_1\theta_2 \\ &\quad + C_7\theta_1\theta_3 + C_8\theta_1\theta_4 + C_9\theta_2\theta_3 + C_{10}\theta_2\theta_4 + C_{11}\theta_3\theta_4 + C_{12}\theta_1 \\ &\quad + C_{13}\theta_2 + C_{14}\theta_3 + C_{15}\theta_4 + C_{16} \end{aligned} \tag{12}$$

where $\theta_i = \lambda_i^2 x + u_0 y - 3/2(\lambda_i^4 + (u_0^3/\lambda_i^2))t$ and where the constants A_i, B_j, C_k can be determined by using the superposition formula (7) and by taking into account the conditions (8) and (9). Furthermore, if $\lambda_3 = \lambda_1^*$, $\lambda_4 = \lambda_2^*$ and $\alpha_i (i = 1\text{–}4)$ are suitably chosen, it is possible to obtain a 'two-lump' solution from f_{1234}.

One can also obtain new solutions of yet a different type by combining the (polynomial) solution f_1 we used earlier on, with a solitary wave generating solution f_2. Namely, choose $f_0 = 1$, $f_1 = \theta_1 \equiv \lambda_1^2 x + u_0 y - 3/2(\lambda_1^4 + (u_0^3/\lambda_1^2))t + \alpha_1$ and $f_2 = 1 + \exp\eta$ where $\eta = px + qy - 1/2(p^3 + q^3 + 3\lambda_2^2 p + 3\lambda_2 p^2 + 3(u_0^2/\lambda_2)q - 3(u_0/\lambda_2)q^2)t + \eta_0$ with p, q, η_0 being constant. Imposing the relation

$$q = \frac{u_0 p}{\lambda_2(\lambda_2 + p)}$$

we have that $f_0 \xrightarrow{\lambda_2} f_2$. In this case, a new function f_{12} can be calculated from the superposition formula (7),

$$\begin{aligned} f_{12} = \frac{p+\lambda_2-\lambda_1}{p+\lambda_1+\lambda_2}\theta_1 \exp\eta - \frac{2\lambda_1^2(p+\lambda_2)}{(p+\lambda_1+\lambda_2)^2}\exp\eta + \frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2}\theta_1 - \frac{2\lambda_1^2\lambda_2}{(\lambda_1+\lambda_2)^2} \\ + G(y,t)\exp[-(\lambda_1+\lambda_2)x] \end{aligned} \tag{13}$$

where $G(y, t)$ is an arbitrary function of y and t. It can be verified that when $G(y, t)$ is chosen as

$$G(y,t) = c_0 \exp\left[\left(\frac{u_0}{\lambda_1} + \frac{u_0}{\lambda_2}\right)y + \frac{1}{2}\left(\lambda_1^3 + \lambda_2^3 - \frac{u_0^3}{\lambda_1^3} - \frac{u_0^3}{\lambda_2^3}\right)t\right] \qquad (c_0 \text{ constant}) \tag{14}$$

the function f_{12} given by expression (13) satisfies the conditions (8) and (9). Thus f_{12} given by (13) and (14) is a new solution of equation (2). Similarly, more solutions may be obtained by use of the superposition formula (7). For example, from the commutation diagram

$$\begin{array}{cccccc} & & & G_{123} & & \\ & & \overset{\lambda_1}{\nearrow} & & \overset{\lambda_4}{\searrow} & \\ G_{23} = f_{12}|_{\substack{\lambda_1\to\lambda_2\\ \lambda_2\to\lambda_3}} & & & & & G_{1234} \\ & & \underset{\lambda_4}{\searrow} & & \underset{\lambda_1}{\nearrow} & \\ & & & G_{234} & & \end{array}$$

where

$$\theta_i \equiv \lambda_i^2 x + u_0 y - \frac{3}{2}\left(\lambda_i^4 + \frac{u_0^3}{\lambda_i^2}\right) t + \alpha_i$$

$$\eta_i = px + \frac{u_0 p}{\lambda_i(\lambda_i + p)} y - \frac{1}{2}\left[p^3 + \frac{u_0^3 p^3}{\lambda_i^3(\lambda_i + p)^3} + 3\lambda_i^2 p + 3\lambda_i p^2 \right.$$
$$\left. + 3\frac{u_0^3 p}{\lambda_i^3(\lambda_i + p)} - 3\frac{u_0^3 p^2}{\lambda_i^3(\lambda_i + p)^2}\right] t + \eta_i^{(0)}$$

with $p, q, \alpha_i, \eta_i^{(0)}$ constant, it will be possible to obtain new solutions G_{123}, G_{234} and subsequently G_{1234}.

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